Polynomial Equations Example

A polynomial equation sets a polynomial equal to zero. Its degree is the greatest exponent, and a degree-three polynomial can have up to three real roots. The rational-root theorem narrows possible rational roots; the factor theorem says P(r) = 0 exactly when (x − r) is a factor. This example finds a root, divides out its factor, and solves the remaining quadratic.

FORMULA
Rational-root theorem: for a rational root p/q in lowest terms, p divides the constant and q divides the leading coefficient. Factor theorem: P(r) = 0 if and only if (x − r) divides P(x).

WORKED EXAMPLE

Solve $x^3 - 6x^2 + 11x - 6 = 0$.

1. List rational-root candidates

The constant is −6 and the leading coefficient is 1, so test the integer factors of 6.

Candidates: ±1, ±2, ±3, ±6; P(1) = 1 − 6 + 11 − 6 = 0

2. Use the factor theorem and divide

Since P(1) = 0, (x − 1) is a factor. Synthetic division gives quotient $x^2 - 5x + 6$ and remainder 0.

1 | 1 −6 11 −6; quotient $x^2 - 5x + 6$; remainder 0

3. Factor the quadratic quotient

Find two numbers with product 6 and sum −5: −2 and −3.

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

4. Set each factor equal to zero

Apply the zero-product property to all three linear factors.

$$(x - 1)(x - 2)(x - 3) = 0; \text{ therefore } x = 1, 2, 3$$

ANSWER

Roots: x = 1, 2, and 3

The cubic factors over the integers into three linear factors, giving three real roots.

CHECK

Expand: $(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$. The coefficients match the original polynomial.